COUNTEREXAMPLES TO LIFTING OF HAMILTONIAN AND CONTACT ISOTOPIES

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Abstract. We construct counterexamples to lifting properties of Hamiltonian and contact isotopies

Introduction

1. Isotopy lifting

1.1. Symplectic reduction along a hypersurface. We recall a notion of a symplectic reduction along a hypersurface [AG]. A smooth hypersurface in a symplectic manifold carries a field of characteristic directions which are kernels of the restriction of the symplectic structure to the hypersurface. We will suppose that a space of characteristics (i.e. integral curves of that field) is a manifold (it always holds locally) and, in particular, that a natural projection of the hypersurface to the space of characteristics is a smooth fibration. The manifold of characteristics descends a natural symplectic structure from the initial symplectic structure [AG]. The resulting symplectic manifold is called a symplectic reduction along the hypersurface.

Let M be a symplectic reduction along a hypersurface $\widetilde{Z}$ in a symplectic manifold $\widetilde{M}$. Denote by $\widetilde{\Pi}$ the natural projection $\widetilde{Z} \to M$. For a subset $\widetilde{L}$ in $\widetilde{M}$ its symplectic reduction along $\widetilde{Z}$ is, by definition, $\widetilde{\Pi}(\widetilde{L} \cap \widetilde{Z})$. It is well known that for a Lagrangian submanifold in $\widetilde{M}$ which is transversal to $\widetilde{Z}$ its symplectic reduction along $\widetilde{Z}$ is an immersed Lagrangian submanifold in M.

It is said that an isotopy $I(t)$ ($t \in [0,1]$) of L (L is a symplectic reduction of $\widetilde{L}$) lifts in $\widetilde{M}$ if there exists an isotopy $\widetilde{I}(t)$ ($t \in [0,1]$) of $\widetilde{M}$ such that $I(t)(L)$ is a symplectic reduction of $\widetilde{I}(t)(\widetilde{L})$.

1.2. Lifting of an isotopy. The following Hamiltonian isotopy lifting property was claimed for the case of general (i.e. not only along a hypersurface) symplectic reduction in [EG] (Lemma 2.5.1):

Statement 1.1. *If the subset $\widetilde{L} \subset \widetilde{M}$ is closed and the projection of $\widetilde{L} \cap \widetilde{Z}$ to M is proper, then every compact Hamiltonian isotopy $I(t)$ in M lifts to a compact Hamiltonian isotopy $\widetilde{I}(t)$ in $\widetilde{M}$ which maps some neighbourhood $\widetilde{Z_0} \subset \widetilde{Z}$ of $\widetilde{L} \cap \widetilde{Z} \subset \widetilde{Z}$ into $\widetilde{Z}$ for all $t \in [0,1]$.*

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If the hypersurface $\widetilde{Z}$ is compact and cooriented then it is easy to prove that statement. Any smooth compactly supported extension $\widetilde{H}_t$ of a peimage of a Hamiltonian H_t of the isotopy on M gives us a required isotopy on $\widetilde{M}$. Surprisingly, in contrast with an analogues statement in the smooth category, that proof could not be adapted to the general non-compact situation and it is indeed wrong. Formal mistake in the proof of 1.1 in [EG] is an assumption that the Hamiltonian vector field of $\widetilde{H}_t$ is globally integrable (i.e. gives rise to a phase flow defined for all values of time) on $\widetilde{Z}$. We show below that in general one need to have a "sufficiently big room" in $\widetilde{M}$ to construct a lifting of a compactly supported isotopy of a reduction of a Lagrangian submanifold.

2. Contact reduction along a hypersurface

We recall the notion of contact reduction along a hypersurface. The following construction is a contact analog of the symplectic reduction along a hypersurface above. Consider a contact manifold $(\widetilde{N}, \widetilde{\xi})$ and a smooth hypersurface $\widetilde{Z}$ in $\widetilde{N}$. We will suppose that at each point of $\widetilde{Z}$ tangent plane to $\widetilde{Z}$ is transversal to ξ. In that case ξ carries a natural (characteristic) field of directions ([AG] ???). Similarly to the symplectic case we suppose that its integral curves forms a manifold N, so that the natural projection $\Pi\colon \widetilde{Z} \to N$ is a smooth fibration. Then N carries a canonical contact structure ξ which is uniquely defined by the condition – preimage of ξ under the action of $d\Pi$ concides with $\widetilde{\xi} \cap T\widetilde{Z}$.

Definitions of the contact reduction of a subset along a hypersurface and lifting of an isotopy coincide with the symplectic case above. The following contact isotopy lifting property was formulated (modulo few misprints) in [EG] (Lemma 2.6.1):

Statement 2.1. *If the subset $\widetilde{L}$ is closed in $\widetilde{N}$ and the projection of $\widetilde{L} \cap \widetilde{Z}$ to N is proper, then for every compact contact isotopy $I(t), t \in [0,1]$ in N there exists a compact contact isotopy $\widetilde{I}(t)$ in $\widetilde{N}$ such that*
(a) $\Pi(\widetilde{I}(t)\widetilde{L} \cap \widetilde{Z}) = I(t)L$,
(b) $\widetilde{I}(t)$ maps some neighbourhood of $\widetilde{L} \cap \widetilde{Z} \subset \widetilde{Z}$ into $\widetilde{Z}$ for all $t \in [0,1]$.

3. Counterexamples

We will show that Statements 2.1 and 2.1 are not true in general.

3.1. Contact case. Let us start from the contact case and describe $\widetilde{N}, \widetilde{Z}$ and $\widetilde{L}$. Let M be a closed connected manifold. Consider the contact manifold $J^1(M \times \mathbb{R}) = T^*M \times T^*\mathbb{R} \times \mathbb{R}$, denote by (q,p) the canonical coordinates on the factor $T^*\mathbb{R}$. Let $\widetilde{N}$ be an open submanifold of $J^1(M \times \mathbb{R})$ given by inequalities $|q| < 1, |p| < 1$. $\widetilde{N}$ is a contact manifold with the induced contact structure. We define $\widetilde{Z}$ to be a hypersurface in $\widetilde{N}$ given by an equation $q = 0$. Consider the 1-jet extension $j^1\widetilde{f} \subset J^1(M \times \mathbb{R})$ of zero function $\widetilde{f}$ on $M \times \mathbb{R}$, let $\widetilde{L}$ be $j^1\widetilde{f} \cap \widetilde{N}$. For such a hypersurface $\widetilde{Z}$ the contact reduction N is naturally contactomorphic to the space $J^1(M)$ with its standard contact structure, L is the 1-jet extension $j^1 f$ of zero function f on M. The hypersurface $\widetilde{Z}$ and Legendrian manifold $j^1 f$ satisfy conditions of Statement

2.1. Note that for any smooth function g on M j^1g could be connected with j^1f by compactly supported contact isotopy.

Statement 3.1. *Consider a compactly supported contact isotopy $\varphi_{t,t\in[0,1]}$ of the manifold $J^1(M)$. Suppose that $\varphi_1(j^1f) = j^1g$ for a function $g\colon M \to \mathbb{R}$. Suppose there exists a compactly supported lifting of that isotopy to a contact isotopy on $\widetilde{N}$ satisfying conditions of Statement 2.1. Then $\max\limits_{x\in M} g(x) \leqslant 1$, $\max\limits_{x\in M} g(x) - \min\limits_{x\in M} g(x) \leqslant 2$.*

As a corollary we get that it is impossible to lift any contact isotopy joining j^1f with j^1g where $\max\limits_{x\in M} g(x) > 1$. Note that this result is non-trivial even M is a point.

Proof. Consider a compactly supported contact isotopy of $\widetilde{N}$ and extend it trivially to the compactly supported contact isotopy $\widetilde{\varphi}_{t,t\in[0,1]}$ (supported in $\widetilde{N}$) of $J^1(M\times\mathbb{R})$. By Chekanov theorem ([Ch]) there exists N such that $\widetilde{\varphi}_1(j^1\widetilde{f})$ is given by quadratic at infinity (with respect to $\mathbb{R}^N$) generating family $F\colon M\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$. Consider now the function F as a family $G_{q,q\in\mathbb{R}}$ of functions depending on $q\in\mathbb{R}$, $\mathbb{R}$ is the factor in $M\times\mathbb{R}\times\mathbb{R}^N$, $G_q(x,w) = F(x,q,w)$ for $x\in M, w\in\mathbb{R}^N$. Recall that Cerf diagramm of family of functions is a graph of all critical values. Consider the Cerf diagramm Γ of G_q. For $|q| \geqslant 1$ all critical values of $G_q(x,w)$ are equal to zero. Cerf diagramm Γ contains a graph of continuous function $h\colon [-1,1]\to\mathbb{R}$, such that $h(0) = \max\limits_{x\in M} g(x)$. Existence of such a function follows from Viterbo's theory of selected values of quadratic at infinity functions. We briefly sketch it here.

For a quadratic at infinity function $G\colon M\times\mathbb{R}^N\to\mathbb{R}$ and non-zero homological class $\alpha\in H_*(M,\mathbb{Z}_2)$ one can correspond a critical value $c(\alpha,G)$ of function G in a following way. Denote by G^a the sublevel set $\{x\in M\times\mathbb{R} | G(x)\leqslant a\}$. For a sufficiently big number $C>0$ the pair (G^C, G^{-C}) is naturally homotopy equivalent to the Thom space of a trivial bundle over M. Denote by T the Thom isomorphism $H_*(M;\mathbb{Z}_2)\to H_*(G^C,G^{-C};\mathbb{Z}_2)$. Then $c(\alpha,G) = \inf\{a | T\alpha\in i_*H_*(G^a,G^{-C};\mathbb{Z}_2)\}$, where i is a natural inclusion map $(G^a,G^{-C})\to(G^C,G^{-C})$. It turns out that $c(\alpha,G)$ is a critical value of G and it depends on G continuously. Consider a function $h\colon\mathbb{R}\to\mathbb{R}$, $h(q) = c([M],G_q)$, $[M]$ is Z_2-fundamental class of M. Consider the number $h(0)$. Note that G_0 is a quadratic at infinity generating family for j^1g. We claim that if G is a quadratic family for 1-jet extension j^1g of a function $g\colon M\to\mathbb{R}$ then $c([M],G) = \max\limits_{x\in M} g(x)$. Fastest way to prove it is to use Theret uniqueness theorem [Th], claiming in that particular case that G is equivalent after stabilization to a quadratic stabilization of g. For stabilizations (i.e. functions of type $g(x)+Q(w)$, Q is nondegenerate quadratic form) that statement is obvious.

The graph of the function h is, by definition, a subset in Γ. Note that $h(q)=0$ for $|q|\geqslant 1$, since all the critical values of G_q are zero. For generic small perturbation $\widetilde{\varphi}^*_{t,t\in[0,1]}$ of $\widetilde{\varphi}_{t,t\in[0,1]}$ supported in $\widetilde{N}$ and the corresponding perturbed family $G^*_{q,q\in\mathbb{R}}$ is sufficiently generic and the corresponding perturbed function h^* is piece-wise smooth. For any $t\in[0,1]$ Legendrian manifold $\widetilde{\varphi}_t(j^1\widetilde{f})$ is transverse to $\widetilde{Z}$. Hence, the reduction along $\widetilde{Z}$ of $\widetilde{\varphi}_1(j^1\widetilde{f})$ is an 1-jet extension of a function g^* which is C_0-close to g. So the number $h^*(0)$ is close to $\max\limits_{x\in M} g(x)$. For generic point $q_0\in]-1,1[$ (except a countable discreet set in $]-1,1[$) in a neighborhood $U(q_0)$ of q_0 critical point corresponding to $c([M], G^*_q)$ smoothly depends on q: $h^*(q) = F^*(x(q), q, w(q))$ for a smooth functions $x(q), w(q)$, such that for any $q \in U(q_0)$ $(x(q), w(q))$ is a critical point of G^*_q. Point $w(q)$ is a critical point of $F^*(x(q), q, .)\colon \mathbb{R}^N \to \mathbb{R}$ and generate a point $l(q) \in \widetilde{\varphi}^*_1(j^1\widetilde{f})$. The absolute value of p-coordinate of $l(q)$ is at most 1, since $l(q) \in \widetilde{N}$. By chain rule the derivative $\frac{dh^*}{dq}(q)$ equals to that p-coordinate. Thus $h^*(0) < 1$, since h^* is a piece-wise smooth continuous function whose derivative is at least -1 and $h^*(1) = 0$.

Since $\max\limits_{x \in M} g(x)$ is close to $h^*(0)$ we get the first inequality of the statement. Similarly, using the class $1 \in H_0(M; \mathbb{Z}_2)$, we get $\min\limits_{x \in M} g(x) \geqslant -1$. Hence, $\max\limits_{x \in M} g(x) - \min\limits_{x \in M} g(x) \leqslant 2$. □

3.2. Symplectic case. A counterexample in symplectic case is similar to the contact case above. Let M be a closed connected manifold of the dimension at least 1. Consider the symplectic manifold $T^*M \times T^*\mathbb{R}$, denote by (q, p) the canonical coordinates on the factor $T^*\mathbb{R}$. Let $\widetilde{N}$ be an open submanifold of $T^*M \times T^*\mathbb{R}$ given by inequalities $|q| < 1, |p| < 1$. $\widetilde{N}$ is a symplectic manifold with the induced contact structure. We define $\widetilde{Z}$ to be a hypersurface in $\widetilde{N}$ given by an equation $q = 0$. Consider the grapf $\Gamma(d\widetilde{f}) \subset T^*M \times T^*\mathbb{R}$ of differential of zero function $\widetilde{f}$ on $M \times \mathbb{R}$, let $\widetilde{L}$ be $\Gamma(d\widetilde{f}) \cap \widetilde{N}$. For such a hypersurface $\widetilde{Z}$ the symplectic reduction N is naturally contactomorphic to the space $T^*(M)$ with its standard symplectic structure, L is the grapf $\Gamma(df) \subset T^*M \times T^*\mathbb{R}$ of zero function on M. The hypersurface $\widetilde{Z}$ and Lagrangian manifold $\Gamma(df)$ satisfy conditions of Statement 1.1.

Statement 3.2. *Consider a compactly supported Hamiltonian isotopy $\psi_{t, t \in [0,1]}$ of the manifold $T^*(M)$. Suppose that $\psi_1(\Gamma(df)) = \Gamma(dg)$ for a function $g\colon M \to \mathbb{R}$. Suppose there exists a compactly supported lifting of that isotopy to a Hamiltonian isotopy on $\widetilde{N}$ satisfying conditions of Statement 2.1. Then $\max\limits_{x \in M} g(x) - \min\limits_{x \in M} g(x) \leqslant 2$.*

Proof. Consider a lifting isotopy $\widetilde{\psi}_t$ on $T^*M \times T^*\mathbb{R}$. We get a family $\{\widetilde{\psi}_t(\Gamma(d\widetilde{f}))\}$ of Lagrangian maifolds. Each of them is an exact Lagrangian manifold, thus we can cover that isotopy by compactly supported isotopy of Legendrian manifolds $\widetilde{\Lambda}_t$ projecting to $\{\widetilde{\psi}_t(\Gamma(d\widetilde{f}))\}$ and coinciding with $j^1\widetilde{f}$ outside $\widetilde{N} \times \mathbb{R}$. The reduction of $\widetilde{\Lambda}_1$ along $\widetilde{Z} \times \mathbb{R}$ is a $j^1(g')$, where g' differs from g by a constant. To the family Λ_t we can apply considerations of Statement 3.1 . Concluding inequality on g' obviously finish the proof. □

Any two graphs of differentials of functions could be joined by compactly supported Hamiltonian isotopy. Hence it is imposiible to lift a Hamiltonian isotopy joining the graph of the differential of zero function with the graph of differential of a function g such that $\max\limits_{x \in M} g(x) - \min\limits_{x \in M} g(x) > 2$. Such a function obviously exists if the dimension of M is not zero.

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